MATH 20D Spring 2023 Lecture 5.

Separable ODE's and Intergrating Factors

Outline

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- You must fill out the **Commencement of Academic Activity Survey** which is available via the *Quizzes* tab in Canvas. Please do this is as soon as possible, but no later than Friday this week.
- You should bring a scientific calculator to both midterms and the final exam. You may also bring a double sided handwritten page of notes.

Contents

Definition

We say that a first order ODE is ***separable*** *if it can be factorized into the form*

$$\frac{dy}{dx} = g(x) \cdot p(y)$$

where g and p are functions of x and y respectively.

- In many cases, we can apply a technique known as **separation of variables** to determine all the solutions to a given separable equation.

Example

Using the method of separation of variables, find all solutions to the ODE

$$\frac{dy}{dx} = -xy.$$

Determine which of your solutions satisfies the initial condition $y(0) = 1$.

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The result of steps 1 & 2 is an ***implicit solution*** *to the ODE of the form*

$$H(y) = G(x) + C$$

where C is an arbitrary constant, $H'(y) = 1/p(y)$, and $G'(x) = g(x)$.

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Step 3: If possible, simplify the **implicit solution** to obtain an **explicit solutions.**

Newton's Law of Cooling

Suppose B is a solid body of temperature $T(t)$ at time t. There exists a constant

$$k_B > 0 \qquad \text{(independent of } T \text{ and } t\text{)}$$

such that if B is placed in an environment with temperature $M(t)$ at time t then

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A 4:00pm a 95°C cup of coffee is served at a cafe where the temperature is a constant 21°C.

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When $M(t)$ is **non-constant**, Newton's Law of Cooling often gives a 1st order linear ODE which is **not separable** e.g. if $M(t) = e^{-t}$ then $T'(t) = k_B(e^{-t} - T(t))$.

Contents

- Recall that a **first order linear ODE** is an equation of the form

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where $a_1(t)$, $a_0(t)$, and $g(t)$ are functions of t.

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where $P(t) = a_0(t)/a_1(t)$ and $Q(t) = g(t)/a_1(t)$.

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Step 3: Integrate the equation $\frac{d}{dt}(\mu(t)y(t)) = \mu(t)Q(t)$ to obtain the solution

$$y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)Q(t)dt + C\right)$$

Example

Find the general solution to the ODE

$$\frac{1}{2}y'(t) + y(t) = e^{-t}$$

via the method of integrating factors.

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Example

Construct a solution to the ODE

$$\frac{dy}{dx} + y = \sqrt{2 + \cos^2(x)}$$

satisfying $y(1) = 4$.